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On Numerical Semigroups with High Embedding Dimension

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We compute the number of elements of a minimal system of generators for the congruence of a numerical semigroup with embedding dimension minus multiplicity equal to zero, one, and two. For symmetric numerical semigroups we study the cases of embedding dimension minus multiplicity equal to one, two, and three.

INTRODUCTION

A numerical semigroup is a subset S of $\mathbb{N}$ closed under addition, containing the zero element and generating $\mathbb{Z}$ as a group. From this definition, one obtains that S has a minimal (with respect to the inclusion) system of generators as a semigroup, $\{n_0 < n_1 < \cdots < n_p\}$, verifying that the greatest common divisor of these generators is one. The number $p + 1$ is usually called the embedding dimension of S and the number n_0 is called the multiplicity of S.

We define the semigroup homomorphism $\varphi\colon \mathbb{N}^{p+1} \to \mathbb{N}$ as

$$\varphi(a_0, a_1, \ldots, a_p) = a_0 n_0 + a_1 n_1 + \cdots + a_p n_p.$$

Let us denote by σ the kernel congruence of φ. Thus, $S \cong \mathbb{N}^{p+1}/\sigma$. Rédei shows in [3] that σ is finitely generated. The scope of this paper is the computation of the number of elements of a system of generators for σ with minimal cardinality. This problem has been already studied for some cases: Herzog studies in [2] this problem for the case $p = 1$ and $p = 2$;

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Bresinsky treats in [1] the case when $p = 3$ and S is symmetric; Rosales studies in [5] the case $p = n_0 - 1$ and in [6] the case $p = n_0 - 2$ and S symmetric. The main purpose of this paper is the study of the cases $p = n_0 - 2$ and $p = n_0 - 3$ for S an arbitrary numerical semigroup and the cases $p = n_0 - 3$, and $p = n_0 - 4$ for S a symmetric numerical semigroup.

In the first section, we obtain that if $p = n_0 - 2$, the number of elements in a system of generators for σ with minimal cardinality is $(n_0 - 1)(n_0 - 2)/2 - 1$ or $(n_0 - 1)(n_0 - 2)/2$. We also determine the number of elements in a minimal presentation for S when $p = n_0 - 3$. In this case, we see that this number is between $(n_0 - 2)(n_0 - 3)/2 - 2$ and $(n_0 - 2)(n_0 - 3)/2$. As an application of the mentioned results, we give a classification of the semigroups with $n_0 \leq 6$ by the number of elements in a system of generators of its associated congruence with minimal cardinality.

In the second section, we study the symmetric case. For $p = n_0 - 2$, the first author showed in [6] that $\#\rho = (n_0 - 1)(n_0 - 2)/2 - 1$. We extend this result for the cases $p = n_0 - 3$ and $p = n_0 - 4$, and we show that the cardinality of a minimal system of generators for the congruence associated to the semigroup is $(n_0 - 2)(n_0 - 3)/2 - 1$ and $(n_0 - 3)(n_0 - 4)/2 - 1$, respectively. As an application we study the cases in which S can be a complete intersection and we compute the number of a minimal system of generators for the congruence associated to a symmetric numerical semigroup with $n_0 \leq 8$.

PRELIMINARIES

In [4], the first author gives an algorithm to compute a system of generators, ρ, for σ with minimal cardinality. From the results given in that paper, it is derived that the concepts of a system of generators for σ with minimal cardinality and a minimal system (with respect to the inclusion) of generators of σ coincide. Next, we give a sketch of this construction.

For every $n \in S$, we define the graph $\mathbf{G}_n = (V_n, E_n)$, as

$$V_n = \{n_i \in \{n_0, \ldots, n_p\} : n - n_i \in S\},$$

$$E_n = \{[n_i, n_j] : n - (n_i + n_j) \in S, i \neq j \in \{0, \ldots, p\}\}.$$

We define ρ_n as

(1) If $\mathbf{G}_n$ is not connected and $\mathbf{G}_n^1 = (V_n^1, E_n^1), \ldots, \mathbf{G}_n^r = (V_n^r, E_n^r)$ are its connected components, then for every $1 \leq i \leq r$ we select an element

$\alpha_i = (a_{0_i}, \ldots, a_{p_i}) \in \mathbb{N}^{p+1} \setminus \{0\}$ such that $\varphi(\alpha_i) = n$ and $a_{k_i} = 0$ for all $n_{k_i} \notin V_n^i$. Define $\rho_n = \{(\alpha_1, \alpha_2), (\alpha_1, \alpha_3), \ldots, (\alpha_1, \alpha_r)\}$.

(2) If $\mathbf{G}_n$ is connected, we define $\rho_n = \emptyset$.

The set $\rho = \bigcup_{n \in \mathbb{N}} \rho_n$ is a system of generators for σ with minimal cardinality (see [4]).

Given $0 \neq n \in S$, we define $S(n)$ to be the Apéry set of n

$$S(n) = \{s \in S: s - n \notin S\}.$$

These sets are finite. In [4], it is shown that if $\mathbf{G}_n$ is not connected, then there exist $s \in S(n_0) \setminus \{0\}$ and $i \in \{1, \ldots, p\}$ such that $n = s + n_i$.

The conductor of a numerical semigroup is the maximum integer not belonging to the semigroup. Such an element always exists, due to the fact that the group generated by a numerical semigroup is $\mathbb{Z}$. It is also known that if x is the greatest element in $S(n_0)$ then the conductor, C, of S is $C = x - n_0$. In [5], the first author studies the relationship of a minimal set of generators for the congruence associated to a semigroup and a minimal set of generators for the congruence associated to the semigroup obtained adding its conductor to the given semigroup, provided that the conductor is greater than n_0. (The case $C < n_0$ only occurs when the semigroup is of the form $\langle n_0, n_0 + 1, \ldots, 2n_0 - 1\rangle$. We may assume that the semigroup under study is not of this type, because this kind of semigroup can be studied using the results appearing in [5].) Let us denote by $S_1 = S \cup \{C\}$. Then, S_1 is generated by $A_1 = \{n_0, \ldots, n_p, C\}$. The set A_1 is a minimal system of generators for S_1 if and only if $x \in S(n_0) \setminus \{0, n_1, \ldots, n_p\}$. If the set A_1 is not a minimal system of generators for S_1, then it can be shown that $\#\rho = \#\rho_1$, where ρ_1 is a minimal system of generators for the congruence associated to S_1. Thus, if we are looking for bounds for $\#\rho$, it does not matter if we study S_1 instead of S. Let A_1' be a minimal system of generators for S_1 and let $S_2 = S_1 \cup \{C_1\}$, with C_1 the conductor of S_1. Then, $S \subset S_1 \subset S_2$ and we can check once more if $A_1' \cup \{C_1\}$ is a minimal generating system for S_2. If it is not so, we construct S_3. We continue this process until the semigroup constructed is minimally generated by the minimal system of generators plus the conductor of the previous semigroup. This procedure must stop, since it can be shown that there is a positive integer m such that $S_m = S^{n_0} = \langle n_0, n_0 + 1, \ldots, 2n_0 - 1\rangle = n_0 + \mathbb{N}$ and in this case $A'_{m-1} \cup \{C_{m-1}\}$ is a minimal system of generators for S_m.

Thus, we can assume that $S = \langle n_0, \ldots, n_p\rangle$ and S_1 is minimally generated by $\{n_0, \ldots, n_p, C\}$. From the results presented by the first author in [5], it is derived that ρ_1 has as many elements as ρ plus $p + 2$ elements (corresponding to the graphs $\mathbf{G}_{C+n_0}, \ldots, \mathbf{G}_{C+n_p}, \mathbf{G}_{2C}$) minus the elements which arise when there is no path from n_0 to n_i in the graphs of the form $\mathbf{G}_{x+n_i}$, where x is the greatest element in $S(n_0)$ and $i \in \{1, \ldots, p\}$.

This idea will enable us to use the results known for semigroups with embedding dimension equal to the multiplicity of the semigroup in order to achieve bounds for the number of elements in minimal presentations of semigroups with embedding dimension their multiplicity minus one. Once this bound is obtained, we can use the same argument to compute a bound for semigroups with embedding dimension their multiplicity minus two. In each step, from the given semigroup we construct a new semigroup being under conditions already studied and then, we count how many elements n_i verify that n_i and n_0 belong to different connected components of $\mathbf{G}_{x+n_i}$, where x is the greatest element in $S(n_0)$.

Finally, we recall the definition of two classes of semigroups. The semigroup S is symmetric if the set $S(n)$ has a maximum with respect to the ordering induced by the addition in S. In this case, the semigroup ring $K[S] = \bigoplus_{s \in S} K y_s$ is Gorenstein. The semigroup S is a complete intersection if $\#\rho$ is exactly p, which is the minimum possible number of elements that a system of generators for σ may contain (see [2]).

1. NUMERICAL SEMIGROUPS WITH HIGH EMBEDDING DIMENSION

For a numerical semigroup, the number of elements in $S(n_0)$ is exactly n_0. Note that all the generators n_i, $i > 0$, are in $S(n_0)$. Thus, the maximal embedding dimension for a semigroup is exactly n_0 (for the zero element is always in $S(n_0)$). The semigroups having maximal embedding dimension have been studied by the first author in [5]. In the mentioned paper, these kinds of semigroups are called MED-semigroups, and it is shown that $\#\rho = n_0(n_0 - 1)/2$. An example of this kind of semigroups is the semigroups of the form $S = \langle n_0, n_0 + 1, \ldots, 2n_0 - 1 \rangle$, which already appeared in the preliminaries.

1.1. *Numerical Semigroups with Embedding Dimension Their Multiplicity Minus One*

In this section, we study the minimal sets of generators for the congruences associated to numerical semigroups, S, such that

$$S(n_0) = \{0, n_1, \ldots, n_p, w\},$$

that is to say, numerical semigroups with embedding dimension $p + 1 = n_0 - 1$.

Note that, since $0 \neq w \in S(n_0)$, w can be written as $w = a_1 n_1 + \cdots + a_p n_p$, with $a_i \neq 0$ for some i. Thus, $w - n_i \in S(n_0)$. Therefore $w = n_i + n_j$ for some $i, j \in \{1, \ldots, p\}$ (not necessarily different).

Let $S_1 = \langle n_0, \ldots, n_p, C \rangle$. As we have seen in the preliminaries, we can assume that S_1 is minimally generated by $\{n_0, \ldots, n_p, C\}$. Note that this is equivalent to the fact that w is the greatest element in $S(n_0)$, because if n_i is the greatest element in $S(n_0)$ for some i, then $n_i = (n_i - n_0) + n_0 = C + n_0$ and therefore $\{n_0, \ldots, n_p, C\}$ does not minimally generate S_1. Hence, S_1 is a MED-semigroup and $\#\rho_1 = n_0(n_0 - 1)/2$. In order to count the elements belonging to ρ, we must count how many generators n_i, $1 \leq i \leq p$ verify that n_i and n_0 are in different connected components of $\mathbf{G}_{w+n_i}$.

Let $w = n_i + n_j$. If n_k and n_0 are not in the same connected component of $\mathbf{G}_{w+n_k}$ for some $k \in \{1, \ldots, p\}$, then $w + n_k - (n_i + n_0) \notin S$ and $w + n_k - (n_j + n_0) \notin S$. This implies that $n_j + n_k, n_i + n_k \in S(n_0)$ which leads to $n_i = n_j = n_k$. Thus, the only graph that can add a new element to ρ is $\mathbf{G}_{w+n_i}$ when $w = 2n_i$ and w cannot be expressed in another way. Let us show that, as a matter of fact, in this case n_i and n_0 are in different connected components of $\mathbf{G}_{w+n_i}$. If it were not so, there would exist a path from n_i to n_0, which would mean that $w + n_i - (n_i + n_k) \in S$ for some $k \neq i$. Nevertheless, $w + n_i - (n_i + n_k) = 2n_i - n_k = w - n_k$ and if $w - n_k \in S$, then w must admit an expression different from $2n_i$, a contradiction.

We get the following theorem:

THEOREM 1. *Let $S = \langle n_0, \ldots, n_p \rangle$ be a numerical semigroup such that $S(n_0) = \{0, n_1, \ldots, n_p, w\}$ and let ρ be a minimal system of generators for the congruence associated to S.*

(1) *If $w = n_i + n_j$ for some $i, j \in \{1, \ldots, p\}$, $i \neq j$ then*

$$\#\rho = \frac{(n_0 - 1)(n_0 - 2)}{2} - 1.$$

(2) *Otherwise,*

$$\#\rho = \frac{(n_0 - 1)(n_0 - 2)}{2}.$$

Proof. Let q be the number of elements in ρ arising from the fact that n_i is not in the same connected component of n_0 in $\mathbf{G}_{w+n_i}$. We have just shown that if $w = n_i + n_j$ with $i \neq j$, then $q = 0$. Otherwise, $q = 1$. This concludes the proof, because $\#\rho_1 = n_0(n_0 - 1)/2 = \#\rho + (p + 2) - q = \#\rho + n_0 - q$ and therefore $\#\rho = (n_0 - 1)(n_0 - 2)/2 - 1 + q$. ∎

As an application of the previous theorem, one can get a classification of semigroups having $n_0 \leq 5$ by the number of elements in a minimal system of generators for its congruence. The only cases which are not complete intersections and are not covered by [2] or [5] are $n_0 = 4$, $p = 2$ and $n_0 = 5$, $p = 3$. These are covered by Theorem 1.

1.2. *Numerical Semigroups with Embedding Dimension Their Multiplicity Minus Two*

In this case,

$$S(n_0) = \{0, n_1, \ldots, n_p, w_1, w_2\}.$$

As in the previous section, we can assume that $S = \langle n_0, \ldots, n_p \rangle$ and that S_1 is minimally generated by $\{n_0, \ldots, n_p, C\}$, that is to say, the greatest element in $S(n_0)$ is in $\{w_1, w_2\}$. We will suppose that w_2 is such a maximum. Hence, $S_1 = \langle n_0, \ldots, n_p, C \rangle$ verifies that $S_1(n_0) = \{0, n_1, \ldots, n_p, C, w_1\}$ and therefore

$$\#\rho_1 \in \left[\frac{(n_0-1)(n_0-2)}{2} - 1, \frac{(n_0-1)(n_0-2)}{2} \right].$$

Thus, we are interested in the elements n_i such that n_0 and n_i are in different connected components of $\mathbf{G}_{w_2+n_i}$. Let q be the number of such elements. We have

$$\#\rho = \#\rho_1 - n_0 + 1 + q.$$

The following cases are possible:

Case 1. $\#V_{w_2} \geq 3$. Take n_i, n_j, n_k as three elements in V_{w_2}. If n_t and n_0 are in a different connected component of $\mathbf{G}_{w_2+n_t}$ then $n_i + n_t, n_j + n_t, n_k + n_t \in S(n_0) \setminus \{0, n_1, \ldots, n_p\}$ (if, for instance, $n_i + n_t - n_0 \in S$, take $n_s \in V_{w_2-n_i}$; then $w_2 + n_t - (n_s + n_0) = (w_2 - n_i - n_s) + (n_i + n_t - n_0) \in S$). But this is impossible, because $\#S(n_0) \setminus \{0, n_1, \ldots, n_p\} = 2$. Thus, in this case $q = 0$ and

(a) If $\#V_{w_1} \geq 2$ then

$$\#\rho = \frac{(n_0-1)(n_0-2)}{2} - 1 - n_0 + 1 = \frac{(n_0-2)(n_0-3)}{2} - 2.$$

(b) If $\#V_{w_1} = 1$ then

$$\#\rho = \frac{(n_0-1)(n_0-2)}{2} - n_0 + 1 = \frac{(n_0-2)(n_0-3)}{2} - 1.$$

Case 2. $\#V_{w_2} = 2$. Let $V_{w_2} = \{n_i, n_j\}$. From the fact that $\#S(n_0) \setminus \{0, n_1, \dots, n_p\} = 2$ it is easily derived that w_2 must be equal to $n_i + n_j$. If n_k is such that there is no path from n_0 to n_k in $\mathbf{G}_{w_2+n_k}$, then $n_i + n_k, n_j + n_k \in S(n_0)$. Hence, $n_i = n_k$ and $w_1 = 2n_i$, or $n_j = n_k$ and $w_1 = 2n_j$. Let us study the different possibilities for $\#V_{w_1}$.

(a) $\#V_{w_1} = 1$. (Note that in this case $w_1 = 2n_t$ for some t and this expression is unique.)

(i) If $w_1 \neq 2n_i$ and $w_1 \neq 2n_j$ then n_k and n_0 are in the same connected component of $\mathbf{G}_{w_2+n_k}$, for all k. Hence, $q = 0$ and

$$\#\rho = \frac{(n_0 - 1)(n_0 - 2)}{2} - n_0 + 1 = \frac{(n_0 - 2)(n_0 - 3)}{2} - 1.$$

(ii) If $w_1 = 2n_i$ then this expression is unique. Hence, $w_2 + n_i - (n_i + n_k) \in S$ if and only if $k \in \{i, j\}$ and $w_2 + n_i - (n_j + n_k) \in S$ if and only if $k = i$, which means that the connected component of n_i in $\mathbf{G}_{w_2+n_i}$ is $\{n_i, n_j\}$. Thus, $q = 1$ and

$$\#\rho = \frac{(n_0 - 1)(n_0 - 2)}{2} - n_0 + 1 + 1 = \frac{(n_0 - 2)(n_0 - 3)}{2}.$$

(iii) If $w_1 = 2n_j$ we obtain the same result.

(b) $\#V_{w_1} \geq 2$.

(i) If $w_1 \neq 2n_i$ and $w_1 \neq 2n_j$ then n_k and n_0 are in the same connected component of $\mathbf{G}_{w_2+n_k}$, for all k. Hence, $q = 0$ and

$$\#\rho = \frac{(n_0 - 1)(n_0 - 2)}{2} - 1 - n_0 + 1 = \frac{(n_0 - 2)(n_0 - 3)}{2} - 2.$$

(ii) If $w_1 = 2n_i$ then, since $\#V_{w_1} \geq 2$, w_1 can be expressed as $n_k + n_l$ with $i \notin \{k, l\}$. Let us show that n_i and n_0 are in the same connected component of $\mathbf{G}_{w_2+n_i}$. Note that $w_2 + n_i - (n_j + n_k) = 2n_i - n_k = n_l \in S$ and that $w_2 + n_i - (n_j + n_l) = 2n_i - n_l = n_k \in S$. Thus, $\{n_i, n_j, n_k, n_l\}$ are in the same connected component of $\mathbf{G}_{w_2+n_i}$. Since $i \notin \{k, l\}$ then $n_j + n_k$ and $n_j + n_l$ are different from w_2. This means that $n_j + n_l \notin S(n_0)$ or $n_j + n_k \notin S(n_0)$ (or both). In the first case, $w_2 + n_i - (n_k + n_0) = n_j + n_l - n_0 \in S$ and in the second, $w_2 + n_i - (n_l + n_0) = n_j + n_k - n_0 \in S$, which implies in both cases that n_0 is connected with n_i. Thus, in this case $q = 0$ and

$$\#\rho = \frac{(n_0 - 1)(n_0 - 2)}{2} - 1 - n_0 + 1 = \frac{(n_0 - 2)(n_0 - 3)}{2} - 2.$$

(iii) If $w_1 = 2n_j$ we get the same result.

Case 3. $\#V_{w_2} = 1$. Then, $w_2 = 2n_i$ or $w_2 = 3n_i$ for an element $i \in \{1, \ldots, p\}$. In both cases, the expression of w_2 is unique and therefore n_i is an isolated point of $\mathbf{G}_{w_2+n_i}$ (otherwise, $w_2 + n_i - (n_i + n_k) = w_2 - n_k \in S$ for some $k \neq i$ which is impossible, because $\#V_{w_2} = 1$). Thus, $q \geq 1$.

(a) $\#V_{w_1} = 1$. If n_k and n_0 are in different connected components of $\mathbf{G}_{w_2+n_k}$, then $n_i + n_k \in S(n_0)$, and this only occurs if $i = k$. Hence, $q = 1$ and

$$\#\rho = \frac{(n_0 - 1)(n_0 - 2)}{2} - n_0 + 1 + 1 = \frac{(n_0 - 2)(n_0 - 3)}{2}.$$

(b) $\#V_{w_1} = 2$. If $w_2 = 3n_i$ then w_1 must be $2n_i$ and this expression must be unique, since $\#V_{w_2} = 1$. But this implies that $\#V_{w_1} = 1$, which is a contradiction. Then, $w_2 = 2n_i$, $w_1 = n_k + n_l$, for some $k \neq l$, and these expressions are unique. If n_t $(t \neq i)$ and n_0 are not connected in $\mathbf{G}_{w_2+n_t}$ then $w_2 + n_t - (n_i + n_0) \notin S$, which means that $n_i + n_t \in S(n_0)$ and therefore $\{i, t\} = \{l, k\}$. Hence, there is no $s \notin \{i, t\}$ such that n_s and n_0 are not connected in $\mathbf{G}_{w_2+n_s}$. Let us show that if $w_1 = n_i + n_t$ then $\{n_i, n_t\}$ is a connected component of $\mathbf{G}_{w_2+n_t}$ and therefore $q = 2$. The element $w_2 + n_t - (n_i + n_s) = w_1 - n_s$ belongs to S if and only if $s \in \{i, t\}$, because w_1 is expressed in an unique way. Besides, $w_2 + n_t - (n_t + n_s) = w_2 - n_s$ is in S only if $s = i$, because $V_{w_2} = \{n_i\}$. Thus,

$$\#\rho = \frac{(n_0 - 1)(n_0 - 2)}{2} - 1 - n_0 + 1 + 2 = \frac{(n_0 - 2)(n_0 - 3)}{2}.$$

(c) $\#V_{w_1} \geq 3$. As before, w_2 must be equal to $2n_i$. If n_j $(j \neq i)$ and n_0 are in different connected components of $\mathbf{G}_{w_2+n_j}$, then $w_2 + n_j - (n_i + n_0) \notin S$ and therefore $n_i + n_j = w_1$. Since $\#V_{w_1} \geq 3$, we can choose $n_k \in V_{w_1} \setminus \{n_i, n_j\}$. Note that $w_2 + n_j = 2n_i + n_j = w_1 + n_i$ and consequently n_i, n_j, and n_k are in the same connected component of $\mathbf{G}_{w_2+n_j}$. Let us show that n_k and n_0 are connected. The element $w_2 + n_j - (n_k + n_0) = w_1 + n_i - (n_k + n_0) = (w_1 - n_k) + n_i - n_0$ belongs to S if and only if $w_1 - n_k + n_i \notin S(n_0)$ (note that $w_1 - n_k \in S$). It is enough to show that $w_1 - n_k + n_i \neq w_1$ and $w_1 - n_k + n_i \neq w_2$. Since $n_i \neq n_k$, $w_1 - n_k + n_i$ cannot be w_1. If $w_1 - n_k + n_i = w_2$, then $w_1 + n_i = 2n_i + n_k$ and therefore $n_i + n_j = w_1 = n_i + n_k$, which means that $n_j = n_k$, and this is not possible. Thus, $q = 1$ and

$$\#\rho = \frac{(n_0 - 1)(n_0 - 2)}{2} - 1 - n_0 + 1 + 1 = \frac{(n_0 - 2)(n_0 - 3)}{2} - 1.$$

We have shown the following theorem:

THEOREM 2. *Let $S = \langle n_0, \dots, n_p \rangle$ be a numerical semigroup such that $S(n_0) = \{0, n_1, \dots, n_p, w_1, w_2\}$ and let ρ be a minimal system of generators for the congruence associated to S. Then*

$$\#\rho \in \left[\frac{p(p+1)}{2} - 2, \frac{p(p+1)}{2}\right]$$
$$= \left[\frac{(n_0-2)(n_0-3)}{2} - 2, \frac{(n_0-2)(n_0-3)}{2}\right].$$

Furthermore, if we assume $w_1 < w_2$, then

- $\#\rho = (n_0-2)(n_0-3)/2 - 2$ *if and only if* $\#V_{w_2} \geq 2$ *and* $\#V_{w_1} \geq 2$.
- $\#\rho = (n_0-2)(n_0-3)/2 - 1$ *if and only if* $\#V_{w_2} \geq 3$ *and* $\#V_{w_1} = 1$, *or* $\#V_{w_2} = 2$, $\#V_{w_1} = 1$, *and* $V_{w_2} \cap V_{w_1} = \emptyset$, *or* $\#V_{w_2} = 1$ *and* $\#V_{w_1} \geq 3$.
- $\#\rho = (n_0-2)(n_0-3)/2$ *if and only if* $\#V_{w_2} = 2$ *and* $V_{w_1} \subset V_{w_2}$, *or* $\#V_{w_2} = 1$ *and* $\#V_{w_1} \leq 2$.

As we did in the previous section for $n_0 \in \{1, \dots, 5\}$, one can now study the case $n_0 = 6$ and see which are the possible values of $\#\rho$. The cases that are not complete intersections and do not lay in the scope of the results in [2] nor in the results appearing in [5] are $p = 3$ and $p = 4$. The former case is covered by Theorem 2 and the latter by Theorem 1.

1.3. *Numerical Semigroups with Embedding Dimension Their Multiplicity Minus Three*

From the results obtained in the previous sections one could think that the bounds for this case are $(n_0-3)(n_0-4)/2 - 3 \leq \#\rho \leq (n_0-3)(n_0-4)/2$. If we take $S = \langle 7, 8, 10, 19 \rangle$, we get $S(7) = \{0, 8, 10, 16, 18, 19, 20\}$ and $\#\rho = 7 > (7-3)(7-4)/2$, and therefore these bounds are not correct. If our computations are correct, the number of elements in ρ is in the interval $[(n_0-3)(n_0-4)/2 - 3, (n_0-3)(n_0-4)/2 + 1]$. We do not include the proof here because of the big amount of cases and subcases arising. This ‘‘bad’’ result makes one think of alternative ways of study for numerical semigroups with not so high embedding dimension.

2. SYMMETRIC NUMERICAL SEMIGROUPS WITH HIGH EMBEDDING DIMENSION

If S is a symmetric semigroup then the set $S(n_0)$ has a maximum with respect to the ordering induced in S by the addition, and this maximum cannot belong to a minimal system of generators for S, but for $n_0 = 2$.

This implies that the maximal embedding dimension is exactly $n_0 - 1$. In [6], the first author denotes this kind of semigroups by MEDSY-semigroups and shows that $\#\rho = (n_0 - 1)(n_0 - 2)/2 - 1$, which coincides with the first case of Theorem 1, but for n_0 equal to two or three. Note that symmetric numerical semigroups always lie in the first case of Theorem 1 because if $S(n_0) = \{0, n_1, \ldots, n_p, w\}$ has a maximum, then w must be this maximum and it must be of the form $w = n_i + n_j$ for some $i \neq j$ (provided $p \geq 2$). If S is a complete intersection, then S is Gorenstein and therefore must lie in the first case. Under this setting, n_0 must be equal to 4 or equal to 1 ($S = \mathbb{N}$). Thus, for $n_0 \geq 5$ there are no complete intersection semigroups with $p = n_0 - 2$.

If $p = n_0 - 3$, and $S(n_0) = \{0, n_1, \ldots, n_p, w_1, w_2\}$ then w_2, the maximum of $S(n_0)$, must be $w_1 + n_i$ for some n_i, which leads to $w_1 = 2n_i$ (if $\#V_{w_1} \geq 2$ then take $n_k \neq n_l \in V_{w_1}$; the elements $n_k + n_i, n_l + n_i \in S(n_0) \setminus \{w_2\}$, which means that $w_1 = n_k + n_i = n_l + n_i$, a contradiction). Since $w_2 - n_j \in S$ for all j, we have that the expression $w_2 = 3n_i$ is not unique, and therefore if $p \geq 3$ (the cases $p = 0$ and $p = 1$ are trivial and the case $p = 2$ has been already studied by Herzog in [2]), then $\#V_{w_2} \geq 3$ which means that

$$\#\rho = \frac{(n_0 - 2)(n_0 - 3)}{2} - 1.$$

Note that if S is a complete intersection and $p = n_0 - 3$, then from the equation $\#\rho = p(p+1)/2 - 1 = p$ we get $n_0 = 5$. Thus, if $n_0 \geq 6$, there are no complete intersection numerical semigroups with their multiplicity equal to their embedding dimension minus two.

Finally, let us study the case $p = n_0 - 4$:

$$S(n_0) = \{0, n_1, \ldots, n_p, w_1, w_2, w_3\}.$$

The following theorem provides a result analogous to the ones obtained so far for $p > n_0 - 4$.

THEOREM 3. *Let $S = \langle n_0, \ldots, n_p \rangle$ be a symmetric numerical semigroup such that $S(n_0) = \{0, n_1, \ldots, n_p, w_1, w_2, w_2\}$, $p \geq 3$, and let ρ be a minimal system of generators for the congruence associated to S. Then*

$$\#\rho = \frac{(n_0 - 3)(n_0 - 4)}{2} - 1.$$

Proof. Note that if S is symmetric then the greatest element x in $S(n_0)$ is in $\{w_1, w_2, w_3\}$, because $x - y \in S$ for all $y \in S(n_0)$. Without loss of generality we can assume that $x = w_3$ and that $w_1 < w_2$. Thus, $S_1 = S \cup$

$\{C\}$ is minimally generated by $\{n_0, \dots, n_p, C\}$ and S_1 is under the conditions of the previous section. As we have done before, we must check how many elements in ρ arise from the fact that n_i and n_0 are in different connected components of $\mathbf{G}_{w_3+n_i}$ for $n_i \in \{1, \dots, p\}$. Assume that n_i, $i \geq 1$, is such that n_i and n_0 are in different connected components of $\mathbf{G}_{w_3+n_i}$. Then, since $w_3 - n_k \in S$ for all $k \in \{1, \dots, p\}$ we get that if n_0 is not connected with n_i in $\mathbf{G}_{w_3+n_i}$, then the elements $n_k + n_i$ are in $S(n_0)$ for all $k \in \{1, \dots, p\}$ (otherwise the edges $[n_i, n_k]$ and $[n_k, n_0]$ are in $E(\mathbf{G}_{w_3+n_i})$). But $p \geq 3$ and $n_1 + n_i, \dots, n_p + n_i$ are all different and non-comparable in S. This is a contradiction with the fact that the maximum number of non-comparable elements in $S(n_0) \setminus \{0, n_1, \dots, n_p\}$ is two. This means that

$$\#\rho = \#\rho_1 - n_0 + 2.$$

If we want to show that $\#\rho = (n_0 - 3)(n_0 - 4)/2 - 1$, then we must prove that $\#\rho_1 = (n_0 - 2)(n_0 - 3)/2$ and this is equivalent to demonstrating (by Theorem 2) that $\#V_{w_2} = 2$ and $V_{w_1} \subset V_{w_2}$, or that $\#V_{w_2} = 1$ and $\#V_{w_1} \leq 2$.

First of all, let us show that $\#V_{w_2} \leq 2$ and that $\#V_{w_1} \leq 2$. If n_i, n_j, n_k are three different elements in V_{w_2} then take $n_l \in V_{w_3 - w_2}$. The elements $n_l + n_i, n_l + n_j, n_l + n_k$ are three different non-comparable (w.r.t. the ordering induced in S by the addition) elements in $S(n_0) \setminus \{0, n_1, \dots, n_p\}$, and this is not possible. In the same way, it is shown that $\#V_{w_1} \leq 2$.

Observe that $w_3 - w_2 \in S(n_0)$, and therefore $w_3 - w_2 = w_2$, $w_3 - w_2 = w_1$ or $w_3 - w_2 \in \{n_1, \dots, n_p\}$. If $w_3 - w_2 = w_2$, then take $n_k \in V_{w_2}$. The element $w_2 + n_k \in S(n_0)$ is greater than w_1 and it is neither w_2 nor w_3, a contradiction. Analogously, we can show that $w_3 - w_2 \neq w_1$. Thus, $w_3 - w_2 \in \{n_1, \dots, n_k\}$.

We have seen that $w_3 - w_1 \neq w_2$ and therefore $w_3 - w_1 = w_1$ or $w_3 - w_1 \in \{n_1, \dots, n_p\}$. If $w_3 - w_1 = w_1$, take $n_k \in V_{w_1}$. Then, $w_1 + n_k \in S(n_0) \setminus \{w_3, w_1\}$ and therefore $w_1 + n_k = w_2$.

Briefing, we get that there are two possible cases:

(1) There exists $i \neq j$ such that $w_3 = w_2 + n_i = w_1 + n_j$.

(2) There exists i, j such that $w_3 = w_2 + n_i$ and $w_2 = w_1 + n_j$.

Let us study each case separately:

(1) It is enough to show that if $\#V_{w_2} = 2$ then $V_{w_1} \subset V_{w_2}$, because if $\#V_{w_2} = 1$ then, since $\#V_{w_1} \leq 2$, we are done. Assume that $\#V_{w_2} = 2$. Then $w_2 = n_s + n_t$ for some $s < t$. As $w_3 = w_2 + n_i$, then $n_s + n_i, n_t + n_i \in S(n_0) \setminus \{w_3\}$. Hence, $n_s + n_i = w_1$ and $n_t + n_i = w_2 = n_s + n_t$, which means that $n_s = n_i$ and consequently $w_1 = 2n_i$. Besides, $w_3 = w_1 + n_j =$

$2n_i + n_j = w_2 + n_i$ and this leads to $w_2 = n_i + n_j(\{i,j\} = \{s,t\})$. Let us assume that w_1 can be expressed in a different way. Then $w_1 = n_k + n_l$ with $i \notin \{k,l\}$ and $k \neq l$. Since $w_3 = w_1 + n_j, n_k + n_j, n_l + n_j \in S(n_0)$. However, the expression $w_2 = n_i + n_j$ is unique, which implies that $w_1 = n_k + n_j = n_l + n_j$, and this is impossible since $k \neq l$. We have shown that $V_{w_1} = \{n_i\} \subset \{n_i, n_j\} = V_{w_2}$.

(2) If we show that $\#V_{w_2} = 1$ then, since $\#V_{w_1} \leq 2$, we are done. If $\#V_{w_2} = 2$, then $w_2 = n_k + n_l$, for some $n_k < n_l$. Hence, $n_k + n_i, n_l + n_i \in S(n_0) \setminus \{w_3\}$, because $w_3 = w_2 + n_i$. This implies that $n_k + n_i = w_1$ and $n_l + n_i = w_2$, but this leads to $n_l + n_i = w_2 = w_1 + n_j = n_k + n_i + n_j$, which means that $n_l = n_k + n_j$, and this is impossible. ∎

The restriction $p \geq 3$ is unimportant, because for the cases $p = 0$ and $p = 1$ the semigroup is a complete intersection, and the case $p = 2$ has been already studied by Herzog for S a numerical semigroup not necessarily symmetric.

With the results obtained in this paper one can compute the possible values of $\#\rho$ for the symmetric numerical semigroups with $n_0 \leq 8$. Observe that if $n_0 \leq 6$, the study can be done using the results exposed in the previous sections together with the results appearing in [2, 5]. Recall that if $n_0 \geq 3$ and the semigroup is symmetric, then $p \leq n_0 - 2$. Note also that, due to the results obtained by Herzog in [2], if $p = 2$ then the semigroup is symmetric if and only if it is a complete intersection. Thus, for $n_0 = 7$ and $p \leq 2$ we get complete intersections; for $p = 4$ we can apply Theorem 3; for $p = 5$ we get a MEDSY-semigroup. Finally, if $n_0 = 8$ and $p \leq 2$ we get once more a complete intersection semigroup; for $p \in \{4, 5, 6\}$ we can use the results exposed in this paper; for $p = 3$, using the results obtained by Bresinsky in [1], we get that $\#\rho \in \{3, 5\}$.

Note that for $n_0 = 8$ and $p = 3 = n_0 - 5$ the statement $\#\rho = (n_0 - 4)(n_0 - 5)/2 - 1$ is false. Taking, for example, $S = \langle 8, 10, 12, 15 \rangle$, one can compute ρ and obtain that $\#\rho = 3 \neq 4 \times 3/2 - 1 = 5$.

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